$$\sum_{k=0}^{n} \int_0^\infty F_k' F'_{n-k} d\eta = \int_0^\infty \int_0^\eta L_{n-1} d\eta d\eta \qquad n \geqq 1 \quad (15)$$

$$\int_0^\infty F_0' \, L_0 \, d\eta = E = \frac{u_1 h}{\nu} \quad (16)$$

$$\sum_{k=0}^{n} \int_0^\infty F_k' L_{n-k} \, d\eta = 0 \qquad n \geqq 1 \quad (17)$$

The boundary conditions are

$$\begin{aligned} &\text{At } \eta = 0: \quad F_n(0) = F_n''(0) = L_n'(0) = 0 \\ &\text{As } \eta \to \infty: \quad F_n'(\infty) = L_n(\infty) \to 0 \end{aligned} \qquad n \geqq 0 \quad (18)$$

The solution F_0 is similar to that for a free jet,

$$F_0 = 3^{2/3} M^{1/3} \tanh\left[\frac{1}{2(3)^{1/3}} M^{1/3} \eta\right] \quad (19)$$

Equation (12) is linear in L_0 and can be integrated to yield

$$L_0 = L_0(0) \exp\left[-\frac{Pr}{3} \int_0^\eta F_0 \, d\eta\right] \quad (20)$$

The surface value $L_0(0)$ may be determined by substituting Eqs. (19) and (20) into Eq. (16). After integrating and simplifying,

$$L_0 = \frac{E}{3^{2/3} M^{1/3} I(Pr)} \left\{\cosh\left[\frac{1}{2(3)^{1/3}} M^{1/3} \eta\right]\right\}^{-2Pr} \quad (21)$$

where

$$I(Pr) = \int_0^\infty (\cosh \xi)^{-2(1+Pr)} \, d\xi \quad (22)$$

Equation (11) is linear in F_n and is uncoupled to Eq. (13); therefore, it can be integrated independently. Equation (15) provides a relationship to determine $F_n'(0)$. Equation (13) is linear in L_n and can be integrated after F_n is solved from Eqs. (11) and (15). Finally, Eq. (17) determines $L_n(0)$. The integration may be carried by analog computers or by numerical methods. The recursion formulas enable the computation of the successive terms in the series of Eqs. (8) and (9).

The forementioned analysis may be extended to the corresponding case in turbulent flow. For this case, the kinematic viscosity ν and the thermal conductivity k are replaced by the eddy kinematic viscosity ϵ and eddy conductivity κ, respectively. Assuming that the eddy thermal diffusivity and the eddy kinematic viscosity are equal and using Prandtl's second hypothesis, one may write

$$\frac{\kappa}{\rho C_p} = \epsilon = \epsilon^* \frac{b u_0}{b^* u_0^*} = \epsilon^* \left(\frac{x}{x^*}\right)^{1/2} = \epsilon^* \left(\frac{X}{X^*}\right)^{1/2} \quad (23)$$

where b is the width of the mixing zone, u_0 is the liquid surface velocity, * is the value at a reference distance x^*, $X = u_1 x/\epsilon^*$, and $X^* = u_1 x^*/\epsilon^*$. According to Ref. (1), $b \sim x$, $u_0 \sim x^{-1/2}$, and hence $b u_0 \sim x^{1/2}$ for a turbulent free jet. Introducing the same dimensionless variables used previously for the laminar case except with ν replaced by ϵ^* and $1/Pr$ replaced by $\kappa/C_p \rho \epsilon^*$, one obtains the following momentum and energy equations, respectively:

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = a \frac{\partial}{\partial X} \int_0^Y \Theta dY + \left(\frac{X}{X^*}\right)^{1/2} \frac{\partial^2 U}{\partial Y^2} \quad (24)$$

$$U \frac{\partial \Theta}{\partial X} + V \frac{\partial \Theta}{\partial Y} = \left(\frac{X}{X^*}\right)^{1/2} \frac{\partial^2 \Theta}{\partial Y^2} \quad (25)$$

The momentum and energy integral equations are the same as for the laminar case.

Again, a similar solution does not exist for the turbulent case. However, a similar solution for turbulent free jets ($a = 0$) exists with $\psi \sim X^{1/2} F(\eta)$, $U \sim X^{-1/2} F'(\eta)$, and $\eta = Y/X$. From Eq. (7), $\Theta \sim X^{-1/2} L(\eta)$ in order to satisfy the requirement that the energy integral is independent of X. For $a \neq 0$, one may consider the following series expansions:

$$F(\eta, aX^c) = \frac{\psi}{X^{1/3}} = F_0(\eta) + (aX^c) F_1(\eta) + (aX^c)^2 F_2(\eta) + \ldots \quad (26)$$

$$L(\eta, aX^c) = \Theta X^{1/2} = L_0(\eta) + (aX^c) L_1(\eta) + (aX^c)^2 L_2(\eta) + \ldots \quad (27)$$

where c is a constant to be determined. Following the same procedure for the laminar case, one obtains exactly Eqs. (10–18), if $c = \frac{3}{2}$, $X^* = \frac{4}{9}$, and $Pr = 1$. Since X^* is an arbitrary reference quantity, it is permissible to use the value $\frac{4}{9}$. Therefore, the solution for the turbulent case may be obtained readily once the solution for the laminar case with $Pr = 1$ is obtained.

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General Asymptotic Suction Solution of the Laminar Compressible Boundary Layer with Heat Transfer

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For very large suction velocities at the wall, the asymptotic solution of the steady compressible two-dimensional laminar boundary layer over a surface of negligible curvature is derived under the following general conditions: an arbitrary prescribed axial pressure gradient, variable suction velocity, an arbitrary prescribed variable wall temperature, variable density, an arbitrary Mach number, a constant but arbitrary Prandtl number, and constant specific heats but variable coefficients of viscosity and, hence, of heat conductivity. It is shown that the dimensionless asymptotic velocity and temperature profiles remain the same regardless of the pressure gradient and of the variability of both the suction velocity and the wall temperature. A Reynolds analogy for this solution is demonstrated. Finally, comparison of the asymptotic solution is made with recent numerical similarity solutions.

Nomenclature

c_p	= specific heat at constant pressure
f_w	= $[2/(m+1)](v_w/u_1)R_x^{1/2}$ (Ref. 8)
k	= coefficient of heat conductivity
L	= characteristic streamwise length
M	= $u/[(\gamma - 1)c_p T]^{1/2}$ = Mach number
Pr	= $\mu c_p/k$ = Prandtl number
R_L	= $\rho_1 u_1 L/\mu_1$ = Reynolds number
R_x	= $\rho_1 u_1 x/\mu_1$ = Reynolds number
T	= absolute temperature

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T_e = equilibrium wall temperature for zero heat transfer
u,v = velocity components in x and y directions, respectively
v_w = (positive) suction velocity at wall
x,y = distance coordinates along and normal to surface, respectively
γ = c_p/c_v = ratio of specific heats
μ = coefficient of viscosity
ρ = mass density
σ,ξ = variables defined by Eq. (5)
ζ = variable defined by Eq. (9)
η = $(y/x)R_x^{1/2}$ (Ref. 8)
θ = $(T - T_w)/(T_1 - T_w)$

Subscripts

1 = local outer edge of boundary layer
w = wall

Griffith and Meredith[1] and (independently) Schlichting[2] obtained the well-known asymptotic suction velocity profile

$$u/u_1 = 1 - e^{-\rho v_w y/\mu} \tag{1}$$

for a uniform suction velocity in incompressible flow over a flat plate (zero pressure gradient). Iglisch[3] subsequently showed, for this case, the actual approach of the velocity profiles to that of Eq. (1) for increasing values of the parameter $(v_w/u_1)R_x^{1/2}$. Pretsch[4] and (independently) Watson[5] showed, for incompressible flow, that, in the limit of very large suction velocities, Eq. (1) remains valid even for arbitrary axial pressure gradients $[u_1 = u_1(x)]$ and variable suction velocities $v_w(x)$. The case of compressible flow was first treated by Young,[6] who obtained the asymptotic suction velocity and temperature profiles for flow over a flat plate with zero heat transfer and arbitrary Mach number. Lew and Fanucci[7] subsequently extended Young's analysis to the case of compressible flow over a flat plate with a prescribed uniform wall temperature. In Refs. 6 and 7 the suction velocity was assumed to be uniform. However, the Prandtl number, though constant, was kept arbitrary whereas the viscosity coefficient was variable. In the present analysis it first will be shown, by a suitable extension of the method of analysis of Pretsch,[4] that the asymptotic suction solutions of Lew and Fanucci remain essentially valid for arbitrary pressure gradients, variable suction velocity $v_w(x)$, and variable wall temperature $T_w(x)$. It is assumed that conditions (e.g., x derivatives) are such that the usual equations of the laminar boundary layer remain valid.

The momentum, continuity, and energy equations are

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \rho_1 u_1 \frac{du_1}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \tag{2}$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{3}$$

$$\rho u c_p \frac{\partial T}{\partial x} + \rho v c_p \frac{\partial T}{\partial y} = -\rho_1 u_1\left(\frac{du_1}{dx}\right) u + \frac{\partial}{\partial y}\left(k \frac{\partial T}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{4}$$

Here, u_1 (together with ρ_1) is considered as a prescribed function of x. To obtain a solution for very large suction, a change of variables will be made from (x,y) to (σ,ξ), where

$$\sigma = x \qquad \xi = [\rho_w(x)v_w(x)/\mu_w(x)]y \tag{5}$$

Such a change of variables is suggested, for example, by Eq. (1). Noting that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \sigma} + \frac{\mu_w}{\rho_w v_w}\,\xi\,\frac{d}{d\sigma}\left(\frac{\rho_w v_w}{\mu_w}\right)\frac{\partial}{\partial \xi}$$
$$\frac{\partial}{\partial y} = \frac{\rho_w v_w}{\mu_w}\frac{\partial}{\partial \xi} \tag{6}$$

and using Eq. (3) to solve for ρv under the condition $\rho v = -\rho_w(x)v_w(x)$ at $y = 0$, Eq. (2) can be written in the following form:

$$\left(\frac{\rho}{\rho_1}\right)\left(\frac{u}{u_1}\right)\left[\frac{\partial u}{\partial(\sigma/L)} + \xi\,\frac{\mu_w}{\rho_w v_w}\,\frac{d}{d(\sigma/L)}\left(\frac{\rho_w v_w}{\mu_w}\right)\frac{\partial u}{\partial \xi}\right] - \left(\frac{\mu_1}{\mu_w}\right)\left(\frac{\rho_w v_w}{\rho_1 u_1}\right)^2 R_L \frac{\partial u}{\partial \xi} - \left(\frac{\partial u}{\partial \xi}\right)\int_0^\xi \left[\frac{1}{\rho_1 u_1}\frac{\partial(\rho u)}{\partial(\sigma/L)} + \xi\left(\frac{\mu_w}{\rho_w v_w}\right)\frac{d}{d(\sigma/L)}\left(\frac{\rho_w v_w}{\mu_w}\right)\frac{\partial}{\partial \xi}\left(\frac{\rho u}{\rho_1 u_1}\right)\right] d\xi = \frac{du_1}{d(\sigma/L)} + \frac{\mu_1}{\mu_w}\left(\frac{\rho_w v_w}{\rho_1 u_1}\right)^2 R_L \frac{\partial}{\partial \xi}\left(\frac{\mu}{\mu_w}\frac{\partial u}{\partial \xi}\right) \tag{7}$$

For the asymptotic suction solution, let the parameter $(\rho_w v_w/\rho_1 u_1)^2 R_L \rightarrow \infty$. Then Eq. (7) reduces to

$$-\frac{\partial u}{\partial \xi} = \frac{\partial}{\partial \xi}\left(\frac{\mu}{\mu_w}\frac{\partial u}{\partial \xi}\right) \tag{8}$$

To solve Eq. (8) for general variable μ, a change of variables from (σ,ξ) to (σ,ζ) may be made, where

$$\zeta = \int_0^\xi \left(\frac{\mu_w}{\mu}\right) d\xi \tag{9}$$

Equation (8) then reduces to

$$(\partial^2 u/\partial \zeta^2) + (\partial u/\partial \zeta) = 0 \tag{10}$$

The solution of Eq. (10), under the boundary conditions $u = 0$ at $\zeta = 0$ and $u \rightarrow u_1(\sigma) = u_1(x)$ as $\zeta \rightarrow \infty$, is

$$u/u_1 = 1 - e^{-\zeta} \tag{11}$$

In terms of the variables (σ,ξ), the energy equation (4), in conjunction with Eq. (3), can be written in the form

$$\left(\frac{\rho}{\rho_1}\right)\left(\frac{u}{u_1}\right) c_p \left[\frac{\partial T}{\partial(\sigma/L)} + \xi\,\frac{\mu_w}{\rho_w v_w}\,\frac{d}{d(\sigma/L)}\left(\frac{\rho_w v_w}{\mu_w}\right)\frac{\partial T}{\partial \xi}\right] - c_p\left(\frac{\mu_1}{\mu_w}\right)\left(\frac{\rho_w v_w}{\rho_1 u_1}\right)^2 R_L \frac{\partial T}{\partial \xi} - c_p\left(\frac{\partial T}{\partial \xi}\right)\int_0^\xi\left[\frac{1}{\rho_1 u_1}\frac{\partial(\rho u)}{\partial(\sigma/L)} + \xi\,\frac{\mu_w}{\rho_w v_w}\,\frac{d}{d(\sigma/L)}\left(\frac{\rho_w v_w}{\mu_w}\right)\frac{\partial}{\partial \xi}\left(\frac{\rho u}{\rho_1 u_1}\right)\right] d\xi = -u\,\frac{du_1}{d(\sigma/L)} + \left(\frac{\rho_w v_w}{\rho_1 u_1}\right)^2 R_L\left(\frac{\mu_1}{\mu_w}\right)\left[\frac{\partial}{\partial \xi}\left(\frac{k}{\mu_w}\frac{\partial T}{\partial \xi}\right) + \frac{\mu}{\mu_w}\left(\frac{\partial u}{\partial \xi}\right)^2\right] \tag{12}$$

Again, for the asymptotic suction solution, let $(\rho_w v_w/\rho_1 u_1)^2 R_L \rightarrow \infty$. Then Eq. (12), in terms of the variables (σ,ζ), reduces to

$$-c_p \frac{\partial T}{\partial \zeta} = \frac{\partial}{\partial \zeta}\left(\frac{k}{\mu}\frac{\partial T}{\partial \zeta}\right) + \left(\frac{\partial u}{\partial \zeta}\right)^2 \tag{13}$$

Equation (13) holds under general variable c_p, μ, and k. Assuming now that c_p is constant and that the Prandtl number $Pr \equiv (\mu/k)c_p$ is also constant, and using Eq. (11), Eq. (13) can be put in the form

$$\frac{1}{Pr}\frac{\partial^2(T/T_1)}{\partial \zeta^2} + \frac{\partial(T/T_1)}{\partial \zeta} = -(\gamma - 1)M_1^2 e^{-2\zeta} \tag{14}$$

The solution of Eq. (14), under the boundary conditions $T \rightarrow T_1(\sigma) = T_1(x)$ as $\zeta \rightarrow \infty$ and $T = T_w(\sigma) = T_w(x)$ at $\zeta = 0$, without difficulty is found to be

$$\frac{T}{T_1} = 1 + \left(\frac{T_w}{T_1} - 1\right)e^{-Pr\zeta} + \frac{(\gamma - 1)M_1^2 Pr}{2(2 - Pr)}\left(e^{-Pr\zeta} - e^{-2\zeta}\right) \tag{15}$$

According to Eq. (9), the transformation to the physical

(x,y) plane can, in general, be made by noting that

$$\xi = \int_0^\zeta \left(\frac{\mu}{\mu_w}\right) d\zeta \tag{16}$$

and substituting μ/μ_w as a function of ζ from its assumed variability with, e.g., T in conjunction with Eq. (15). Equations (11, 15, and 16) constitute an asymptotic suction solution for compressible flows with heat transfer for arbitrary $u_1(x)$, $v_w(x)$, $T_w(x)$, constant Pr, and arbitrary variable μ.

In view of the general conditions under which it is valid, the solution derived here is remarkably simple. First, in the special case of low-speed flows ($M_1 \approx 0$) with constant fluid properties† [but still variable $v_w(x)$ and $T_w(x)$], Eqs. (11) and (15) each reduce to a form quite similar to Eq. (1):

$$u/u_1(x) = 1 - e^{-\xi} \qquad \theta = 1 - e^{-Pr\xi} \tag{17}$$

where $\xi = \rho v_w(x) y/\mu$. Second, by setting $(\partial T/\partial \zeta)_w = 0$ and solving for T_w, Eq. (15) is found to imply that the equilibrium wall temperature T_e for zero heat transfer will be

$$T_e/T_1 = 1 + [(\gamma - 1)/2]M_1^2 \tag{18}$$

for all Prandtl numbers. Finally, it is noted that, in general, the local skin-friction coefficient and Nusselt number will be

$$c_f \equiv \frac{(\mu \partial u/\partial y)_w}{(\frac{1}{2})\rho_1 u_1^2} = 2\left(\frac{\rho_w v_w}{\rho_1 u_1}\right) \tag{19}$$

$$Nu \equiv \frac{(k\partial T/\partial y)_w x}{k_1(T_e - T_w)} = Pr\left(\frac{k_w}{k_1}\right)\left(\frac{\rho_w v_w x}{\mu_w}\right) \tag{20}$$

Thus, the Nusselt number here is directly proportional to Pr, in contrast to the case of an impermeable wall, for which Nu is a considerably more complicated function of Pr, depending somewhat on the pressure gradient and varying, for example, approximately as $Pr^{1/3}$ in a zero pressure gradient for $Pr \geq 0.6$. Equations (19) and (20) imply the following type of Reynolds analogy‡:

$$(Nu/c_f)R_x^{-1} = Pr/2 \tag{21}$$

The solutions obtained here may be regarded as approximate solutions of the laminar boundary layer for cases of *finite* but large suction velocities. From this point of view it may be of interest to compare these solutions with the similarity solutions recently obtained numerically by Koh and Hartnett[8] for low-speed flows with constant fluid properties, in which $u_1 \sim x^m$, $v_w(x) \sim x^{(m-1)/2}$, $T_w(x) - T_1 \sim x^n$, $Pr = 0.73$, and $m = 0, \frac{1}{3}, 1$; $n = -1$ to 10. In the notation of Ref. 8, Eqs. (17, 19, and 20) for these flows can be written as

$$u/u_1 = 1 - e^{-[(m+1)/2]f_w\eta} \qquad \theta = 1 - e^{-[Pr(m+1)/2]f_w\eta} \tag{22}$$

$$c_f R_x^{1/2} = (m+1)f_w \qquad Nu/R_x^{1/2} = Pr[(m+1)/2]f_w \tag{23}$$

A detailed comparison has shown that the velocity profiles of Eq. (22) virtually coincide with the profiles obtained in Ref. 8 for suction velocities corresponding to $f_w \geq 8$, while for $f_w = 6$ the difference throughout is within 3%. Moreover, for constant wall temperature ($n = 0$), the temperature profiles of Eq. (22) virtually coincide throughout with those of Ref. 8 for $f_w = 8$. For variable wall temperature, one readily can see from Figs. 3–5 of Ref. 8 the actual approach, with increasing suction, of the temperature profiles to the asymptotic suction profile by comparing the set of profiles for $n = -1$ to 10, corresponding to $f_w = 0$ and $f_w = 1$, with the set corresponding to $f_w = 8$. Whereas the effect of n on the temperature profiles is quite considerable in the case of zero suction ($f_w = 0$), the profiles for $n = -1$ to 10 are seen to come relatively close together (though not yet quite coincident), i.e., they tend to become independent of n, when $f_w = 8$.§

† That is, constant ρ, c_p, μ, and k. The temperature T, however, is permitted to vary.

‡ The relation $k_w/\mu_w = k_1/\mu_1$, following from the assumed constancy of Pr and c_p, is used here.

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§ An approximate solution for large suction, showing analytically the approach of the temperature profiles to the asymptotic profile for these flows, can be obtained by putting $f' = 1, f = f_w$ into Eq. (9) of Ref. 8 and solving for θ. The following solution thus is obtained for $n \geq 0$: $\theta = 1 - e^{-\alpha\eta}$, where

$$\alpha = \frac{Pr(m+1)}{4} f_w + \left[\left(\frac{Pr(m+1)}{4} f_w\right)^2 + nPr\right]^{1/2}$$

As $f_w \to \infty$, this solution approaches that in Eq. (22).

Effects of Controlled Roughness on Boundary-Layer Transition at a Mach Number of 6.0

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The importance of the effects of boundary-layer transition on the aerodynamic characteristics and the heat transfer to a vehicle in flight at hypersonic speeds is well known. As had been discussed in Ref. 1, the considerable scatter found in the data currently available on boundary-layer transition may be due in part to the method of detecting transition. In this investigation of hypersonic boundary-layer transition on a smooth flat plate with a sharp leading edge (leading edge thickness less than 0.002 in.) and with or without controlled roughness, the heat flow rate method of detecting transition has been employed with considerable success. By quickly injecting the model from a sheltered position outside the tunnel wall into an established Mach 6.0 freestream, lateral conduction in the model skin has been kept to a minimum. The tunnel stagnation temperature

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